1. Counting problems

To read:
[1]: 1.2. Sets, 1.3. Number of subsets, 1.5. Sequences, 1.6. Permutations, 1.7. Number of The Number of Ordered Subsets, 1.8. The Number of Subsets of a Given Size, 3.1. The Binomial Theorem, 3.2. Distributing Presents, 3.5. Pascal's Triangle, 3.6. Identities in Pascal's Triangle. [3], Chapters 3.1-3.3.

1.1. Basic results on counting sets.

Notation. Let A be a finite set. We denote by $|A|$ the *cardinality* of A, i. e. the number of elements in the set.

Definition 1.1. Denote by $[n]$ the set of first n natural numbers: $[n] := \{1, 2, \dots n\}$.

Theorem 1.2. *If there exists a bijection between finite sets A and B then $|A| = |B|$.*

Theorem 1.3. *(Addition rule) Let A and B be finite sets. If $A \cap B = \emptyset$ then $|A \cup B| = |A| + |B|$.*

Theorem 1.4. *(Product rule) Let A and B be finite sets. Then*

$$|A \times B| = |A| \cdot |B|.$$

Recall the following formulas:

Proposition 1.5. *The number of functions from $[m]$ to $[n]$ is n^m. This is the number of m-letter words in an n-letter alphabet.*

Proposition 1.6. *The number of permutations of a set of n elements is $n!$*

Proof. This is likely to be familiar to you, but at any rate it follows from the multiplication rule. Call the elements $1, \dots, n$. A permutation can send 1 to any of n elements. Then 2 to any of the $n - 1$ elements remaining, since 1 and 2 cannot be sent to the same. Each step leaves one less option at the next step, for a total of

$$n \times (n - 1) \times \dots \times 2 \times 1$$

permutations. This is $n!$ by definition (or really, if we refuse to skip steps, by induction). □

Proposition 1.7. *The number of ways in which one can choose k objects out of n distinct objects, assuming the order of the elements matters, is $\frac{n!}{(n-k)!}$.*

Proof. It will dramatically speed up computations to note that

$$\frac{n!}{(n-k)!} = n(n-1)\dots(n-k+1)$$

This should be calculated as a product of k numbers, not a ratio of two factorials. In fact, this form also shows how to deduce the formula from the multiplication rule. One has n choices for the first object, then $n - 1$ for the second, culminating in $n - k + 1$ for the last of the k objects.

Notice that when $k = n$, Propositions 1.6 and 1.7 agree. This would be clear even without the explicit formulae: an ordered choice of all n out of the n objects is simply a way to permute them.

Set-theoretically, $n(n-1)\cdots(n-k+1)$ is also the number of injective functions from $[k]$ to $[n]$. □

Proposition 1.8. *The number of ways in which one can choose k objects out of n distinct objects, assuming the order of the elements does not matter, is $\frac{n!}{(n-k)!k!} =: \binom{n}{k}$. This is the same as the number of subsets of k elements of an n-element set.*

Definition 1.9. The numbers $\binom{n}{k} = \frac{n!}{(n-k)!k!}$ are called *binomial coefficients.*

Proof. We already know the number of ordered subsets, by Proposition 1.7. On the other hand, an ordered subset can be obtained in two steps: choose a subset, and then order it. Once the choice of k elements is made, Proposition 1.6 tells us there are $k!$ ways to do the ordering. By the multiplication rule,

$$\frac{n!}{(n-k)!} = \binom{n}{k} k!$$

and we complete the proof by solving for $\binom{n}{k}$.

□

As with unordered choices, there is no need to compute all the factorials. Instead, note that

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{n(n-1)\dots(n-k+1)}{k!}$$

If k is small, then we can afford to compute $k!$ in the denominator. If k is large, then it is better to exploit a basic symmetry of the binomial coefficients.

Proposition 1.10.

$$\binom{n}{k} = \binom{n}{n-k}$$

We will be convenient for us to use the following notation:

Notation. Let A be a finite set and k be a nonnegative integer. Then $\binom{A}{k}$ is the set of k-element subsets of A. We have $\left|\binom{A}{k}\right| = \binom{|A|}{k}$.

1.2. **Binomial coefficients.** The following is called Pascal's triangle

Row	
0	$\binom{0}{0} = 1$
1	$\binom{1}{0} = 1$ $\binom{1}{1} = 1$
2	$\binom{2}{0} = 1$ $\binom{2}{1} = 2$ $\binom{2}{2} = 1$
3	$\binom{3}{0} = 1$ $\binom{3}{1} = 3$ $\binom{3}{2} = 3$ $\binom{3}{3} = 1$
4	$\binom{4}{0} = 1$ $\binom{4}{1} = 4$ $\binom{4}{2} = 6$ $\binom{4}{3} = 4$ $\binom{4}{4} = 1$
5	$\binom{5}{0} = 1$ $\binom{5}{1} = 5$ $\binom{5}{2} = 10$ $\binom{5}{3} = 10$ $\binom{5}{4} = 5$ $\binom{5}{5} = 1$

Proposition 1.11. *The following identities hold:*

(1) $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$.
(2) $\binom{n}{k}$ *is the k-th element in the n-th line of Pascal's triangle.*

Proof. Recall that $\binom{n+1}{k+1}$ is the number of subsets of cardinality $k+1$ in the set $[n+1]$. Each subset of $[n+1]$ either contains the element $n+1$ or not. The number of elements in $\binom{[n+1]}{k+1}$ containing $n+1$ is $\binom{n}{k}$ and the number of elements in $\binom{[n+1]}{k+1}$ *not* containing $n+1$ is $\binom{n}{k+1}$. Now we apply the Addition rule and finish the proof. □

Proposition 1.12. *The number of subsets of an n-element set is 2^n, since we have*

$$2^n = \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n}.$$

The number of subsets of an n-element set having odd cardinality is 2^{n-1}. The number of subsets of an n-element set having even cardinality is 2^{n-1}.

The equalities above can be obtained using the binomial theorem.

Theorem 1.13.

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \ldots + \binom{n}{n}x^n = \sum_{i=0}^{n} \binom{n}{i} x^i.$$

Proof. To prove the binomial theorem, consider how to distribute the multiplication in

$$(1+x)^n = (1+x)(1+x)\ldots(1+x)$$

From each factor $1+x$, we can choose either the 1 or the x to form a product with the other terms. This product is x^k provided we choose x in k out of the n factors. There are $\binom{n}{k}$ such choices, and collecting terms gives the sum $\sum_k \binom{n}{k} x^k$ as claimed. □

Proof of Proposition 1.12. For $x = 1$, respectively $x = -1$, we obtain

$$2^n = \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = \sum_{i=0}^{n} \binom{n}{i}$$

$$0 = \binom{n}{0} - \binom{n}{1} + \ldots + (-1)^n \binom{n}{n} = \sum_{i=0}^{n} (-1)^i \binom{n}{i}.$$

Adding, respectively subtracting the two relations, and dividing each by two, one obtains

$$2^{n-1} = \binom{n}{0} + \binom{n}{2} + \ldots$$

$$2^{n-1} = \binom{n}{1} + \binom{n}{3} + \ldots$$

which proves the statements about the number of even/odd sets.
□

Proposition 1.14. *Assume we have k identical objects and n different persons. Then, the number of ways in which one can distribute this k objects among the n persons equals*

$$\binom{n+k-1}{n-1} = \binom{n+k-1}{k}.$$

Equivalently, it is a number of solutions of the equation $x_1 + \ldots + x_n = k$ in nonnegative integers or the number of k-multisets containing elements from $[n]$. If $k \geq n$ and each persons receives at least 1 object, then the number of possible ways to distribute is $\binom{k-1}{n-1}$.

Proof. Let $\mathcal{A}$ be the set of all solutions of the equation

(1) $$x_1 + \ldots + x_n = k, x_i \in \mathbb{Z}_{\geq 0}.$$

Let $\mathcal{B}$ be the set of all subsets of cardinality $n-1$ in $[k+n-1]$. We construct a bijection $\psi : \mathcal{A} \to \mathcal{B}$ in the following way: a solution $(x_1, \ldots, x_n)$ is mapped to the subset

$$B := \{x_1 + 1, x_1 + x_2 + 2, \ldots, x_1 + x_2 + \ldots + x_{n-1} + n - 1\}.$$

First, we check that B belongs to $\mathcal{B}$. Indeed, the inequalities

$$1 \leq x_1 + 1 < x_1 + x_2 + 2 < \cdots < x_1 + x_2 + \ldots x_{n-1} + n - 1 \leq k + n - 1$$

imply that the elements of B are distinct and belong to $[k+n-1]$.

Next, to show that ψ is a bijection we compute its inverse map. Let B be an element of $\mathcal{B}$. Suppose that

$$1 \leq b_1 < b_2 < \cdots < b_{n-1} \leq k + n - 1$$

are the elements of B written in the increasing order. Then the preimage $\psi^{-1}(B)$ is an n-tuple of integers $(x_1, \ldots, x_n)$ defined by

$$\begin{aligned} x_1 &= b_1 - 1 \\ x_i &= b_i - b_{i-1} - 1, \quad i = 2, \ldots, n-1 \\ x_n &= k + n - 1 - b_{n-1}. \end{aligned}$$

It is easy to see from these equations that the numbers $x_i, i = 1, \ldots n$, are non-negative integers and $x_1 + \ldots + x_n = k$.

Since there is a bijection between sets $\mathcal{A}$ and $\mathcal{B}$, their cardinalities are equal and

$$|\mathcal{A}| = |\mathcal{B}| = \binom{k+n-1}{n-1}.$$

□

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References

[1] Discrete Mathematics (L. Lovasz, J. Pelikan , K. Vesztergombi);
[2] Combinatorics: Set Systems, Hypergraphs, Families of Vectors and Combinatorial Probability (B. Bollobas);
[3] Invitation to Discrete Mathematics (J. Matousek, J. Nesetril).
[4] Extremal combinatorics (S. Jukna).
[5] Combinatorial theory (M. Hall), Blaisdell publishing company, 1967.